# Homework 4 Solutions

1.13 #1 (a) "$\langle x, y\rangle = \sum_{i=1}^{n} x_i |y_i|$."

This formula fails Axiom 1. For example if $n = 2$, $x = (-1,-1)$ and $y = (1,1)$ then
$$\langle x, y\rangle = -1\cdot|1| + (-1)\cdot|1| = -2$$
and
$$\langle y, x\rangle = 1\cdot|-1| + 1\cdot|-1| = 2.$$

This formula also fails Axiom 2. For example if $n = 2$, $x = (1,1)$, $y = (1,-1)$, and $z = (4,1)$ then

$$\langle x, y+z\rangle = 1\cdot|1+4| + 1\cdot|1+(-1)| = 4$$

and
$$\langle x, y\rangle + \langle x, z\rangle = 1\cdot|1| + 1|-1| + 1\cdot|4| + 1\cdot|1| = 7.$$

This formula also fails Axiom 4.

For example, if $n = 2$, and $x = (1, -1)$,

then
$$\begin{aligned} \langle x, x \rangle &= 1 \cdot |1| + (-1) \cdot |-1| \\ &= 0, \end{aligned}$$

but $x$ is not the zero vector.

1.13 #1 (c) " $\langle x, y\rangle = \sum_{i=1}^{n} x_i \sum_{j=1}^{n} y_j$ ".

This formula fails Axiom 4. For example, let $n=2$ and $x=(1,-1)$.

Then $\langle x, x\rangle = (1+(-1))(1+(-1))$

$= 0$,

but $x$ is not the zero vector.

3. "Prove $\langle x, y \rangle = 0$ if and only if $\|x+y\| = \|x-y\|$."

($\Rightarrow$) Suppose $\langle x, y \rangle = 0$.

Then
$$\|x+y\| = \sqrt{\langle x+y, x+y \rangle}$$
$$= \sqrt{\langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle}$$
$$= \sqrt{\langle x, x \rangle + \langle y, y \rangle}$$

Since $\langle x, y \rangle = 0$.

Moreover

$$\|x-y\| = \sqrt{\langle x-y, x-y \rangle}$$
$$= \sqrt{\langle x, x \rangle - 2\langle x, y \rangle + \langle y, y \rangle}$$
$$= \sqrt{\langle x, x \rangle + \langle y, y \rangle}$$

Since $\langle x, y \rangle = 0$.

Thus $\|x+y\| = \|x-y\|$.

($\Leftarrow$) Now suppose $\|x+y\| = \|x-y\|$.

Then
$$\|x+y\| = \|x-y\|$$
$$\Rightarrow \|x+y\|^2 = \|x-y\|^2$$

$$\Rightarrow \langle x+y, x+y\rangle = \langle x-y, x-y\rangle$$

$$\Rightarrow \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle = \langle x,x\rangle - 2\langle x,y\rangle + \langle y,y\rangle$$

$$\Rightarrow \quad \langle x,y\rangle = -\langle x,y\rangle .$$

Since $\langle x,y\rangle$ is a real number and $\langle x,y\rangle = -\langle x,y\rangle$, we must have $\langle x,y\rangle = 0$.

1.13 #7 "If $x$ and $y$ are nonzero elements making an angle $\theta$ with each other, then $\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\cdot\|y\|\cos\theta$."

Now $\|x-y\|^2 = \langle x-y, x-y\rangle$

$$= \langle x,x\rangle - 2\langle x,y\rangle + \langle y,y\rangle$$

$$= \|x\|^2 + \|y\|^2 - 2\langle x,y\rangle. \qquad (*)$$

By definition, $\cos\theta = \dfrac{\langle x,y\rangle}{\|x\|\cdot\|y\|}$,

thus $\langle x,y\rangle = \|x\|\cdot\|y\|\cos\theta$, and substituting this into $(*)$, we get

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\cdot\|y\|\cos\theta$$

as desired.

1.13 #8 " Define

$$\langle f, g\rangle = \int_1^e (\log x) f(x) g(x)\, dx ."$$

☆ Note that $\log x$ is the natural log.

(a) "If $f(x) = \sqrt{x}$, compute $\|f\|$."

Now

$$\|f\|^2 = \langle f, f\rangle$$

$$= \int_1^e (\log x)(\sqrt{x})(\sqrt{x})\, dx$$

$$= \int_1^e x \log x \, dx$$

$$= \frac{1}{4} x^2 (2\log(x) - 1) \Big|_1^e$$

$$= \frac{e^2}{2} - \frac{e^2}{4} - 0 + \frac{1}{2}$$

$$= \frac{e^2}{4} + \frac{1}{2} .$$

So $\|f\| = \sqrt{\frac{e^2}{4} + \frac{1}{2}}$ .

(b) "Find a linear polynomial $g(x) = a + bx$ that is orthogonal to the constant function $f(x) = 1$."

A linear polynomial $g(x)$ is orthogonal to $f(x) = 1$ if

$$\langle f, g \rangle = \int_1^e (\log x)\, g(x)\, dx$$

$$= 0.$$

So $\displaystyle\int_1^e (\log x)(a + bx)\, dx$

$$= a\int_1^e \log x\, dx + b\int_1^e x \log x\, dx$$

$$= a\, x(\log x - 1)\Big|_1^e + \frac{b}{4} x^2 (2\log x - 1)\Big|_1^e$$

$$= ae - ae + a + \frac{be^2}{2} - \frac{be^2}{4} + \frac{b}{4}$$

$$= a + \frac{be^2}{4} + \frac{b}{4}.$$

Thus $g(x) = a + bx$ is orthogonal to $f(x) = 1$ if $a + \frac{be^2}{4} + \frac{b}{4} = 0$.

For example $g(x) = x - \frac{e^2}{4} - \frac{1}{4}$ is orthogonal to $f(x) = 1$.

1.13 #12 (a) "$\langle f, g\rangle = f(1)g(1)$."

This formula violates Axiom 4. For example if $f(x) = x-1$, then

$$\langle f, f\rangle = (1-1)^2$$
$$= 0$$

but $f(x) = x-1$ is not the zero function.

(b) "$\langle f, g\rangle = \left|\int_0^1 f(t)g(t)\,dt\right|$."

This formula violates Axiom 2.
Suppose $f(t) = g(t) = 1$ and $h(t) = -1$.
Then

$$\langle f, g+h\rangle = \left|\int_0^1 1(1+(-1))\,dt\right| = 0$$

but

$$\langle f, g\rangle + \langle f, h\rangle = \left|\int_0^1 1\cdot 1\,dt\right| + \left|\int_0^1 1(-1)\,dt\right|$$
$$= 2.$$

16. (a) "Prove that

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + \langle x,y\rangle + \langle y,x\rangle$$

in every Euclidean space."

Now

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\ &= \|x\|^2 + \|y\|^2 + \langle x,y\rangle + \langle y,x\rangle.\end{aligned}$$